7.1 Boundary value problem

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad 0 \le y \le 10 \qquad \alpha = 1$$

$$T(y=0,t)=0 \qquad T(y=10,t)=45 \qquad T(y,t=0)=20$$

$$T = f(y)\cdot g(t)$$

$$\frac{fg' = \alpha f'' g}{fg} \qquad \frac{g'}{\alpha g} = \frac{f''}{f} = -\lambda^2$$

$$g = c_1 \exp(-\alpha\lambda^2 t) \qquad f = A\sin\lambda y + B\cos\lambda y$$

$B = 0$ since $T(0,t)=0$ and ultimately $T = 4.5y$

$$\Rightarrow \quad T = 4.5y + A\exp(-\alpha\lambda^2 t)\sin\lambda y$$

for $y=10$, $T=45$ $\Rightarrow$ $\sin 10\lambda = 0$

so $\lambda = \frac{n\pi}{10}$ $\quad T = 4.5y + \sum_{n=1}^{\infty} A_n \exp(-\lambda_n^2 t)\sin\lambda_n y$

Initially $(t=0)$, $T = 20$

$$20 - 4.5y = \sum_{n=1}^{\infty} A_n \sin\lambda_n y \quad \left\{ \begin{array}{l}\text{half-range Fourier}\\ \text{sine series}\end{array}\right.$$

$$A_n = \frac{2}{10}\int_0^{10} (20 - 4.5y)\sin\frac{n\pi y}{10}\,dy$$

$$A_n = \frac{50}{n\pi}\cos n\pi + \frac{40}{n\pi}$$

t	y	T
½	9	27.93
"	7	20.07
"	5	20.00
"	3	19.95
"	1	13.65
2	9	35.42
"	7	23.33
"	5	20.06
"	3	17.34
"	1	7.66

7.2 Boundary value problem

$$\frac{\partial T}{\partial t} = 2\frac{\partial^2 T}{\partial y^2} \qquad 0 \le y \le 10$$

$$T(y=0, t) = 20 \qquad T(y=10, t) = 10$$

initially: $T(y, t=0) = 5+5y$

$T = f(y)\cdot g(t)$ as before resulting in

$$T = 20 - y + A\exp(-2\lambda^2 t)\sin\lambda y$$

for $y = 10$, $\sin\lambda y = 0$, so $\lambda = \frac{n\pi}{10}$

$$T = 20 - y + \sum_{n=1}^{\infty} A_n \exp(-2\lambda_n^2 t)\sin\lambda_n y$$

At $t = 0$, $T = 5+5y$, so

$$-15 + 6y = \sum_{n=1}^{\infty} A_n \sin\lambda_n y \qquad \text{therefore,}$$

$$A_n = \frac{2}{10}\int_0^{10} (-15+6y)\sin\lambda_n y \cdot dy$$

So, $A_n = -\frac{90}{n\pi}\cos n\pi - \frac{30}{n\pi}$

y	$t = 1/16$ $T(y,t)$	$t = 1/2$	$t = 8$
1	10.68	17.19	20.20
3	20.00	20.51	20.15
5	30.00	29.99	18.94
7	40.00	38.47	16.22
9	47.95	28.42	12.24