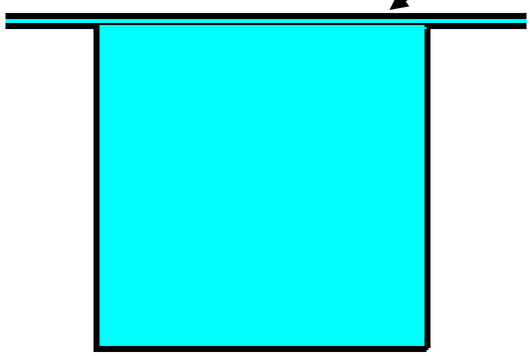
Chemical Engineering Analysis 1

ChE 735 — Final Semester Exercise — Fall 2019

December 9, 2019

Final semester exercise, submission deadline 12/19/2019, 12 noon. Strictly individual effort expected and adequate explanation of your work is essential.

Consider a square cavity completely filled with a viscous fluid initially at rest. At t=0, the upper surface (the lid) begins to slide across the cavity (from left to right) at a constant velocity, V_{TOP}. The situation of interest is depicted here:



We want to use vorticity transport to model this phenomenon. For this exercise, we *assume* that the Reynolds number is small enough such that the convective transport of vorticity can be neglected; i.e., we need only concern ourselves with viscous (or molecular) transport:

$$\frac{\partial \omega}{\partial t} = \nu \left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right].$$

This is a parabolic partial differential equation and we can solve it explicitly by forward-marching in time. The vorticity and the stream function (ψ) are related by

$$-\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.$$

This is an elliptic (Poisson-type) partial differential equation and we can solve it iteratively using Gauss-Seidel or preferably, successive over-relaxation (SOR). We now will construct the necessary logic to solve this problem, in detail, and step-by-step. We place the origin in the lower left-hand corner.

1) We assign an arbitrary value to the stream function on the *entire* boundary of the cavity; we choose ψ=0 for convenience.

2) Since vorticity is neither created nor destroyed in the interior of a homogeneous fluid, we realize that the motion of the upper surface will create velocity gradients, and hence, vorticity. We must adjust the stream function near the top of the cavity to reflect the constant velocity, V_{TOP}. Let the subscript j indicate the top (lid) position; we must assign the correct value to ψ for the uppermost *interior* row:

$$\psi_{i,j-1} = \frac{1}{18}(11\psi_{i,j} + 9\psi_{i,j-2} - 2\psi_{i,j-3} - 6V_{top}\Delta y) .$$

You may recognize that we have used a third-order backward difference to obtain ψ on the top *interior* (j-1) row (the boundary of course has been specified, ψ=0).

3) We can now compute vorticity across the top of the cavity, e.g.,

$$\omega_{i,j} = -\frac{1}{(\Delta y)^2}(\psi_{i,j} - 2\psi_{i,j-1} + \psi_{i,j-2}) .$$

This expression comes directly from the fact that we only have the x-component of velocity at the very top of the cavity:

$$\omega = -\frac{\partial v_x}{\partial y} = -\frac{\partial^2 \psi}{\partial y^2} .$$

The same reasoning applies at the bottom too, but in that case we will use a second-order forward difference approximation. One must also make sure that the no-slip condition is enforced at the bottom and the vertical sides of the cavity in anticipation that we might want to examine larger Reynolds numbers. For the bottom this means $\frac{\partial \psi}{\partial y} = 0$, and for the side walls, $\frac{\partial \psi}{\partial x} = 0$.

4) We calculate the vorticity on both vertical walls, recognizing that only the y-component of velocity need be taken into account. Let the subscript i represent the index for the left vertical wall:

$$\omega_{i,j} = -\frac{1}{(\Delta x)^2}(\psi_{i,j} - 2\psi_{i+1,j} + \psi_{i+2,j}) ,$$

where $\psi_{i,j}$ =0. We treat the right-hand vertical wall in exactly the same manner, but we employ a second-order *backward* difference.

5) Now that we have obtained vorticity values on all four cavity walls, we can compute updated vorticity in the *interior* of the cavity explicitly:

$$\omega_{i,j,k+1} = \nu\Delta t\left[\frac{\omega_{i+1,j,k} - 2\omega_{i,j,k} + \omega_{i-1,j,k}}{(\Delta x)^2} + \frac{\omega_{i,j+1,k} - 2\omega_{i,j,k} + \omega_{i,j-1,k}}{(\Delta y)^2}\right] + \omega_{i,j,k} .$$

We only need two values for the time subscript, k, since we will discard the old vorticity values once the new ones are computed.

6) We use the updated vorticity distribution to calculate the stream function iteratively everywhere in the interior of the cavity:

$$\psi_{i,j} = \psi_{i,j} + \frac{W_{SOR}}{4}(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j} + (\Delta x)^2 \omega_{i,j,k+1}) .$$

W_{SOR} is the relaxation parameter and for a problem like this it should have a value of roughly 1.8 to 1.9. The actual number of iterations required here will depend upon other parameters selected for a particular problem but it is normally modest, possibly on the order of 100. This completes the sequence of calculations for one time-step; we have advanced from t to $t+\Delta t$. We now return to 2) and repeat the entire process.

Project Submission Requirements:

i) Complete description of your logic and solution procedure.

ii) Plot(s) revealing the evolution of the cavity vortex with time.

iii) Quantitative estimate for V_X ($x=L/2,y$) under near steady-state conditions

Problem parameters (all cgs):

$L=5$ $H=5$ $\nu=0.05$ $V_{TOP}=1/20$

Consequently, $\mathrm{Re} = \dfrac{LV_{TOP}}{\nu} = \dfrac{(5)(1/20)}{(0.05)} = 5$